PSEUDO-CANONICAL FORMS AND INVARIANTS OF SYSTEMS OF PARTIAL DIFFERENTIAL EQUATIONS*

BY

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1. Introduction

The theory of invariants of the linear ordinary differential equation dates from 1862, when Sir James Cockle† obtained certain seminvariants. In later papers he made progress in the theory, while other writers,‡ notably Laguerre, Brioschi, Halphen and Forsyth, extended his results. The theory thus developed presents a striking analogy to the theory of algebraic invariants. One point of similarity, with which this paper has much to do, is the fact that by means of a transformation of the type $y = \lambda(x)\bar{y}$, the linear differential equation

$$y^{(n)} + p_1 ny^{(n-1)} + \cdots + p_n y = 0$$

may be made to assume a canonical form, in which the coefficient of $y^{(n-1)}$ is zero. The remaining coefficients are then unchanged when the equation is subjected to any transformation of the type in question, i.e., they are absolute seminvariants. The seminvariants thus obtained are moreover a fundamental set, in the sense that any seminvariant whatever is a function of them and of their derivatives.

The work of Wilczynski§ has shown how the previous theory may be extended to include completely integrable systems of linear homogeneous partial

* Presented to the Society, September 2, 1919.

† *The Correlations of Analysis*, Philosophical Magazine, ser. 4, vol. 24, pp. 531–534.

‡ In the first pages of an article "*Invariants of the General Linear Differential Equation and their Relation to the Theory of Continuous Groups*," American Journal of Mathematics, vol. 21 (1899), pp. 25–84, Bouton gives a brief sketch of these contributions.

§ *Projective Differential Geometry of Curved Surfaces*, these Transactions, vol. 8 (1907), pp. 233–260.

One-Parameter Families and Nets of Plane Curves, these Transactions, vol. 12 (1911), pp. 473–510.

Sur la Théorie Générale des Congruences, Mémoires de l'Académie Royale de Belgique, Classe des Sciences, ser. 2, vol. 3 (1910–1912).

differential equations. In various papers, he and others* have obtained canonical forms of certain such systems, making use of the transformation $y = \lambda(u, v)\bar{y}$. As in the theories referred to, the coefficients of a canonical form are a fundamental set of seminvariants. Green† has proved that for a very large class of completely integrable systems, such a canonical form can always be obtained, and that its coefficients will have the usual properties. Finally should be mentioned a paper of Wilczynski,‡ in which is given a general proof covering all known theories of invariants, as well as many others.

It is characteristic of all the work just referred to that the actual existence of a canonical form has been considered necessary to the proof that the coefficients of such a form are seminvariants. This has resulted, in so far as partial differential equations are concerned, in a failure to obtain canonical forms characterized by the vanishing of certain *coefficients*.§ Instead, certain *functions of the coefficients* have been made to vanish. The seminvariants secured, therefore, have not been as simple as might be desired. The author || has shown that, for a large class of completely integrable systems, a fundamental set of simpler seminvariants can be constructed as the coefficients of a *pseudo-canonical form*. These non-existent pseudo-canonical forms are characterized by the vanishing of certain coefficients. The proof used seemed more complicated than necessary, and one of the aims of the present paper is to furnish a simpler one.

A second purpose is the extension of the theory presented in the previous article to include semi-covariants, invariants and covariants.

Attention will be confined to partial differential equations with a single dependent variable, although the facts exhibited will hold for systems having two or more dependent variables, provided the transformations used are of the type discussed in this paper. It must be admitted, however, that such

* W. W. Denton, *Projective Differential Geometry of Developable Surfaces*, these Transactions, vol. 14 (1913), pp. 175–208.

G. M. Green, *Projective Differential Geometry of Triple Systems of Surfaces*, Columbia Dissertation, 1913.

G. M. Green, *Projective Differential Geometry of One-Parameter Families of Space Curves, and Conjugate Systems of Curves on a Surface*, American Journal of Mathematics, vol. 37 (1915), pp. 215–246.

† *Linear Dependence of Functions of Several Variables, and Completely Integrable Systems of Homogeneous Linear Partial Differential Equations*, these Transactions, vol. 17 (1916), pp. 483–516.

‡ *Invariants and Canonical Forms*, Proceedings of the National Academy of Sciences, vol. 4 (1918), pp. 300–305.

§ The one exception to this statement is the theory of curved surfaces, discussed by Wilczynski in the paper referred to.

|| *Note on Seminvariants of Systems of Partial Differential Equations*, American Journal of Mathematics, vol. 41 (1919), pp. 123–132.

transformations (in cases which involve two dependent variables) are not always the ones of greatest interest geometrically.*

Ordinary differential equations will not be considered, for the reason that the methods of this paper, insofar as they apply to ordinary differential equations, reduce to old methods in such cases.

Particular attention must be called to a change in terminology from the author's paper cited. What were there called *pseudo-canonical forms* are now termed *pseudo-semi-canonical forms*, the first name being reserved for certain new unique forms.

The author wishes to thank Professor Wilczynski for valuable criticisms, which have resulted in an improvement in the form of this paper. He also desires to make the following further acknowledgments: Dr. A. L. Miller, in setting up a completely integrable system of equations to be used as a basis for a special theory, accidentally discovered that seminvariant coefficients resulted in that special case when a certain unallowable transformation was employed. Acting upon this hint, the author succeeded in establishing the results contained in his previous paper. While considering the feasibility of the extensions embraced in the present paper, special results of Dr. W. W. Denton in the theory of developable surfaces were of assistance.

2. Pseudo-semi-canonical forms and seminvariants

Let us assume that the completely integrable system of differential equations (a) has one dependent variable, y, and n independent variables, $u_1, \cdots, u_n$; that it consists of p equations, each of which expresses a certain derivative of y linearly and homogeneously in terms of certain q primary derivatives (including y itself); that no primary derivative is of higher order than any of the left members of the equations (a); that if a given y-derivative,

$$\frac{\partial^{l_1+\cdots+l_n} y}{\partial u_1^{l_1} \cdots \partial u_n^{l_n}},$$

occurs in any equation of (a), then all derivatives of lower order, from which the given derivative may be obtained by differentiation, are also present in that equation.†

We shall group the various y-derivatives and coefficients of (a) as follows:

* See, for example, Wilczynski's basis for the study of congruences of straight lines, in his prize memoir, "*Sur la Théorie Générale des Congruences*," l. c. Compare with that used by Green, in section 8 of his paper, "*Projective Differential Geometry of One-parameter Families of Space Curves etc.*," l. c. The former employs the transformations used in the present paper, while the latter does not.

† The purpose of this assumption is to insure that a transformation of the dependent variable shall replace (a) by a system of exactly the same form. Cf. Green, *Linear Dependence of Functions of Several Variables*, etc., l. c., section 6.

A y-derivative of the same order as the left member of its equation is said to be of the zeroth class. A y-derivative of order i less than the left member of its equation is of the ith class. A coefficient of an ith class y-derivative is of the ith class.

The equation

$$y = \lambda(u_1, \cdots, u_n)\bar{y} \tag{1a}$$

yields by differentiation

$$y_{u_i} = \lambda \bar{y}_{u_i} + \lambda_{u_i}\bar{y} \qquad (i = 1, \cdots, n),$$

$$\cdots\cdots\cdots\cdots\cdots \tag{1b}$$

$$y_{l_1, \cdots, l_n} = \sum \binom{l_1}{p_1} \cdots \binom{l_n}{p_n} \lambda_{p_1, \cdots, p_n} \bar{y}_{l_1 - p_1, \cdots, l_n - p_n},$$

$p_1 = 0, \cdots, l_1, p_2 = 0, \cdots, l_2, \cdots, p_n = 0, \cdots, l_n$, where

$$y_{l_1, \cdots, l_n} = \frac{\partial^{l_1 + \cdots + l_n} y}{\partial u_1^{l_1} \cdots \partial u_n^{l_n}}, \qquad \binom{l_i}{p_i} = \frac{l_i!}{(l_i - p_i)!\, p_i!}.$$

When equations (1) are substituted in the system (a) there results a new system, $(\bar{a})$, of the same form as (a), in which the independent variables are, as before, $u_1 . \cdots . u_n$, while the dependent variables are $\bar{y}, \bar{y}_{u_i}$ $(i = 1, \cdots, n)$, etc. These new variables may be expressed explicitly in terms of y, y_{u_i}, etc., as follows:

$$\bar{y} = \mu y, \qquad \mu = 1/\lambda, \tag{2a}$$

$$\bar{y}_{u_i} = \mu \left[y_{u_i} + \frac{\mu_{u_i}}{\mu} \cdot y \right] \qquad (i = 1, \cdots, n),$$

$$\cdots\cdots\cdots\cdots\cdots, \tag{2b}$$

$$\bar{y}_{l_1, \cdots, l_n} = \mu \sum \binom{l_1}{p_1} \cdots \binom{l_n}{p_n} \frac{\mu_{p_1, \cdots, p_n}}{\mu} \cdot y_{l_1 - p_1, \cdots, l_n - p_n},$$

with $p_1, p_2, \cdots, p_n$ as in (1b).

A coefficient of $(\bar{a})$ of the first class is of the type

$$\bar{\beta} = \beta + \sum_{i=1}^{n} p_i \alpha_i \frac{\mu_{u_i}}{\mu}, \tag{3}$$

where the p_i are integers (including zero) and the α_i are seminvariant zeroth class coefficients of (a).*

Let us take† n equations of the type $\bar{\beta} = 0$, choosing the coefficients β in such a way as to permit the solution of this set of n equations for μ_{u_i}/μ $(i = 1,$

* Cf. A. L. Nelson, l. c., section 2. It was there tacitly assumed that a first-class coefficient would be changed (if at all), by the transformation $(1a)$, by the addition of a multiple of a single such μ-fraction. The structure of higher class coefficients of $(\bar{a})$ is also discussed in this section.

† The possibility of this will be discussed in section 8.

$\cdots, n)$, regarding these fractions as independent unknowns. We denote these solutions by (μ_{u_i}/μ).

It is of course impossible to find a function, $\mu(u_1, \cdots, u_n)$, which will satisfy the n differential equations

$$\bar{\beta}_i = 0, \qquad (i = 1, \cdots, n),$$

unless certain integrability conditions are satisfied. These conditions are

$$\frac{\partial}{\partial u_i}\left(\frac{\mu_{u_j}}{\mu}\right) = \frac{\partial}{\partial u_j}\left(\frac{\mu_{u_i}}{\mu}\right) \qquad (i, j = 1, \cdots, n). \tag{4}$$

Nevertheless, without assuming that equations (4) hold, we substitute the solutions (μ_{u_i}/μ) for μ_{u_i}/μ in all the coefficients of $(\bar{a})$.

However, since (μ_{u_i}/μ) and (μ_{u_j}/μ) $(i, j = 1, \cdots, n;\ i \neq j)$, are two distinct functions of the coefficients of (a), there are two independent ways of substituting for $\mu_{u_i u_j}/\mu$, namely,

$$\frac{\mu_{u_i u_j}}{\mu} = \left(\frac{\mu_{u_i}}{\mu}\right)_{u_j} + \left(\frac{\mu_{u_i}}{\mu}\right)\left(\frac{\mu_{u_j}}{\mu}\right),$$

and

$$\frac{\mu_{u_i u_j}}{\mu} = \left(\frac{\mu_{u_j}}{\mu}\right)_{u_i} + \left(\frac{\mu_{u_i}}{\mu}\right)\left(\frac{\mu_{u_j}}{\mu}\right).$$

This fact would give rise to a lack of uniqueness, which must be avoided by observing the following rule: *For any particular coefficient of* $(\bar{a})$, *we must decide which of the two possible substitutions for* $\mu_{u_i u_j}/\mu$ *is to be used. Throughout the coefficient, that substitution must be used, the symbolic identities* (4) *being employed, if necessary.* For example, suppose that a certain coefficient of $(\bar{a})$ is of the form

$$\cdots + a\frac{\mu_{u_1 u_2}}{\mu} + \cdots + b\frac{\mu_{u_1 u_1 u_2}}{\mu} + c\frac{\mu_{u_1 u_2 u_2}}{\mu} + \cdots$$

If $\mu_{u_1}/\mu = a'$, $\mu_{u_2}/\mu = b'$, then $\mu_{u_1 u_2}/\mu = a'_{u_2} + a'b'$, or $\mu_{u_1 u_2}/\mu = b'_{u_1} + a'b'$. Either substitution for $\mu_{u_1 u_2}/\mu$ may be chosen, but when chosen, must be used for $\mu_{u_1 u_1 u_2}/\mu$, $\mu_{u_1 u_2 u_2}/\mu$, and all other μ-fractions in the coefficient whose numerators are partial derivatives of $\mu_{u_1 u_2}$. That is, supposing that $\mu_{u_1 u_2}/\mu$ is chosen equal to $a'_{u_2} + a'b'$, b'_{u_1} must be replaced by a'_{u_2}, whenever b'_{u_1} or one of its derivatives appears in the coefficient.

Having observed this precaution, it is evident that we have formally obtained a unique form of the completely integrable system (a). We shall call this form a *pseudo-semi-canonical form* (A), of (a), and shall use capital letters for its coefficients. It is completely characterized by the n equations $B_i = 0$, $(i = 1, \cdots, n)$. The seminvariance of the coefficients of (A) may be established by the following argument.

Let us refer to the set of coefficients of the system (a) as $[a]$. Under the transformation $(2a)$ they are replaced by the set $[a_\mu]$, the first class coefficients of which are of the type (3). $[a]$ and $[a_\mu]$ may be regarded as the coefficients of any two systems of type (a) which are equivalent under the transformation $(2a)$. Apply to (a) and (a_μ) the transformation $y = \nu\bar{y}$, which is of the type $(2a)$. The coefficients $[a_\nu]$ and $[a_{\mu\nu}]$ which result have the following relations: (i) $[a_{\mu\nu}]$ are the same functions of $[a_\mu]$ and ν_{u_i}/ν, $(i = 1, \cdots, n)$, as $[a_\nu]$ are of $[a]$ and ν_{u_i}/ν; (ii) $[a_{\mu\nu}]$ are the same functions of $[a]$ and $(\mu\nu)_{u_i}/\mu\nu$, $(i = 1, \cdots, n)$, as $[a_\nu]$ are of $[a]$ and ν_{u_i}/ν.

In $[a_\nu]$ make n coefficients of the first class (see (3)) vanish by a suitable choice of ν_{u_i}/ν, $(i = 1, \cdots, n)$, as functions of $[a]$, regarding these functions (denoted by (ν_{u_i}/ν)), as independent unknowns. If we substitute (ν_{u_i}/ν) for ν_{u_i}/ν throughout $[a_\nu]$, and observe the above mentioned precaution concerning cross-derivatives, we obtain unique functions $[A]$ of the original coefficients $[a]$. The functions $[A]$ are also unique as a set.

Make the corresponding first class coefficients of $[a_{\mu\nu}]$ vanish. This will be accomplished by taking ν_{u_i}/ν equal to the same functions, $(\nu_{u_i}/\nu)'$, of $[a_\mu]$ as the (ν_{u_i}/ν) are of $[a]$. But this choice of the ν_{u_i}/ν is equivalent to taking $(\mu\nu)_{u_i}/\mu\nu$ equal to the same functions, $((\mu\nu)_{u_i}/\mu\nu)$, of $[a]$, so that the functions $((\mu\nu)_{u_i}/\mu\nu)$ and (ν_{u_i}/ν) are identical. The substitution of $(\nu_{u_i}/\nu)'$ for ν_{u_i}/ν throughout $[a_{\mu\nu}]$ will yield a set of unique functions, $[A]'$, of the coefficients $[a]$.

By virtue of the relation (i), the functions $[A]'$ are the same functions of $[a_\mu]$ and $(\nu_{u_i}/\nu)'$ as $[A]$ are of $[a]$ and (ν_{u_i}/ν). But $(\nu_{u_i}/\nu)'$ are the same functions of the coefficients $[a_\mu]$ as (ν_{u_i}/ν) are of the original coefficients $[a]$. Hence the functions $[A]'$ are the same functions of $[a_\mu]$ as $[A]$ are of $[a]$, where $[a_\mu]$ and $[a]$ are the coefficients of any two systems of the type (a) equivalent under the transformation $(2a)$.

In view of the relation (ii), the functions $[A]'$ are the same functions of $[a]$ and $((\mu\nu)_{u_i}/\mu\nu)$ as $[A]$ are of $[a]$ and (ν_{u_i}/ν). Since $((\mu\nu)_{u_i}/\mu\nu)$ and (ν_{u_i}/ν) are identical, we see that the functions $[A]'$ and $[A]$ are identical. Hence they are seminvariants.

We note that the number of seminvariant coefficients of a pseudo-semi-canonical form is $pq - n$, exactly the number of seminvariants in a fundamental set as obtained by Green.* They are moreover independent, no two arising from the same coefficient of (a). Hence *the coefficients* $[A]$ *are a fundamental set of seminvariants.*

3. Semi-covariants

The transformation $(1a)$ leads us to two important sets of equations. One set is that by means of which the new coefficients (those of $(\bar{a})$) are expressed

* *Linear Dependence of Functions*, etc., l. c., section 6.

in terms of the old ones (those of (a)) and of the transformation function μ. The other is the set of equations (2), which express the new variables in terms of the old ones and of μ. There is a parity existing between these two sets which, apparently, has not usually been recognized.

To exhibit one aspect of this parity, let us make in equations (2) the substitutions $\mu_{u_i}/\mu = (\mu_{u_i}/\mu)$, $(i = 1, \cdots, n)$, which are determined by the equations $\bar{\beta}_i = 0$, $(i = 1, \cdots, n)$. In these substitutions, we disregard the factor μ. *The functions*

$$(5) \qquad Y = y, \qquad Y_{u_i} = y_{u_i} + (\mu_{u_i}/\mu)y, \qquad (i = 1, \cdots, n),$$

are relative covariants. We shall prove this proposition by an argument parallel to that made in section 2 for the seminvariance of the functions $[A]$.

When the transformation (2a) is made, the set of variables

$$[y]: \quad y, \, y_{u_i}, \text{ etc.} \qquad (i = 1, \cdots, n),$$

is replaced by the set $[y_\mu]$, given by equations (2). When we apply the transformation $\bar{y} = \nu y$ to the systems (a) and (a_μ), we get the new sets of variables $[y_\nu]$ and $[y_{\mu\nu}]$, respectively. If we substitute $\nu_{u_i}/\nu = (\nu_{u_i}/\nu)$ in $[y_\nu]$, there results (if we disregard the factor ν) the unique set of functions $[Y]$. Similarly, the substitution of $\nu_{u_i}/\nu = (\nu_{u_i}/\nu)'$ in $[y_{\mu\nu}]$ gives rise to the set of functions $[Y]'$. As in section 2, we can prove the following statements, which suffice to establish the truth of the proposition: (i) The set $[Y]'$ are the same functions (neglecting a factor) of $[y_\mu]$ and $[a_\mu]$ as $[Y]$ are of $[y]$ and $[a]$; (ii) Except for a factor, the set $[Y]'$ are the same functions of $[y]$ and $((\mu\nu)_{u_i}/\mu\nu)$ as $[Y]$ are of $[y]$ and (ν_{u_i}/ν). Hence the functions $[Y]'$ and $[Y]$ are identical, except for a factor.

An induction proof may be made as follows: We first show that

$$\overline{\left(\frac{\mu_{u_i}}{\mu}\right)} = \left(\frac{\mu_{u_i}}{\mu}\right) - \frac{\mu_{u_i}}{\mu} \qquad (i = 1, \cdots, n).$$

In order to do this, let us refer to the definitions of (μ_{u_i}/μ). Since these functions arise from n equations of the type (3), we have

$$(6) \qquad B_g = \beta_g + \sum_{i=1}^{n} p_{gi}\, \alpha_{gi} \left(\frac{\mu_{u_i}}{\mu}\right) = 0 \qquad (g = 1, \cdots, n).$$

But, since the combinations B_g are seminvariants,

$$\bar{\beta}_g + \sum_{i=1}^{n} p_{gi}\, \alpha_{gi} \overline{\left(\frac{\mu_{u_i}}{\mu}\right)} = 0 \qquad (g = 1, \cdots, n),$$

the $p_{gi}\, \alpha_{gi}$ being also seminvariants. By use of (3) and (6), we may put these equations in the form

$$\sum_{i=1}^{n} p_{gi}\, \alpha_{gi} \left[\overline{\left(\frac{\mu_{u_i}}{\mu}\right)} - \left(\frac{\mu_{u_i}}{\mu}\right) + \frac{\mu_{u_i}}{\mu} \right] = 0 \qquad (g = 1, \cdots, n).$$

The determinant of this system does not vanish, since we have assumed that the equations (6) are solvable for the (μ_{u_i}/μ). Therefore,

$$\overline{\left(\frac{\mu_{u_i}}{\mu}\right)} = \left(\frac{\mu_{u_i}}{\mu}\right) - \frac{\mu_{u_i}}{\mu} \qquad (i = 1, \cdots, n).$$

As a result of this lemma, we have

$$\bar{Y}_{u_i} = \bar{y}_{u_i} + \overline{\left(\frac{\mu_{u_i}}{\mu}\right)}\bar{y} = \mu\left[y_{u_i} + \frac{\mu_{u_i}}{\mu}y\right] + \left[\left(\frac{\mu_{u_i}}{\mu}\right) - \frac{\mu_{u_i}}{\mu}\right]\mu y = \mu Y_{u_i} \qquad (i = 1, \cdots, n).$$

It is also evident from (2) that $\bar{Y} = \mu Y$. Now assume that

$$\bar{Y}_{l_1, \cdots, l_n} = \mu Y_{l_1, \cdots, l_n}. \tag{7}$$

From the law of formation of the functions (5), it follows that

$$\begin{aligned} &Y_{l_1, \cdots, l_{i-1}, l_i+1, l_{i+1}, \cdots, l_n} \\ &= \sum \binom{l_1}{p_1} \cdots \binom{l_n}{p_n} \left\{ \left(\frac{\mu_{l_1, \cdots, l_{i-1}, l_i+1, l_{i+1}, \cdots, l_n}}{\mu}\right) y_{l_1-p_1, \cdots, l_n-p_n} \right. \\ &\qquad \left. + \left(\frac{\mu_{p_1, \cdots, p_n}}{\mu}\right) y_{l_1-p_1, \cdots, l_{i-1}-p_{i-1}, l_i-p_i+1, l_{i+1}-p_{i+1}, \cdots, l_n-p_n} \right\} \\ &= \sum \binom{l_1}{p_1} \cdots \binom{l_n}{p_n} \left\{ \left[\left(\frac{\mu_{l_1, \cdots, l_n}}{\mu}\right)_{u_i} \right.\right. \\ &\qquad \left. + \left(\frac{\mu_{u_i}}{\mu}\right)\left(\frac{\mu_{l_1, \cdots, l_n}}{\mu}\right) \right] y_{l_1-p_1, \cdots, l_n-p_n} \\ &\qquad \left. + \left(\frac{\mu_{p_1, \cdots, p_n}}{\mu}\right) y_{l_1-p_1, \cdots, l_{i-1}-p_{i-1}, l_i-p_i+1, l_{i+1}-p_{i+1}, \cdots, l_n-p_n} \right\} \\ &= \frac{\partial}{\partial u_i} \sum \binom{l_1}{p_1} \cdots \binom{l_n}{p_n} \left(\frac{\mu_{l_1, \cdots, l_n}}{\mu}\right) y_{l_1-p_1, \cdots, l_n-p_n} \\ &\qquad + \left(\frac{\mu_{u_i}}{\mu}\right) \sum \binom{l_1}{p_1} \cdots \binom{l_n}{p_n} \left(\frac{\mu_{l_1, \cdots, l_n}}{\mu}\right) y_{l_1-p_1, \cdots, l_n-p_n} \\ &= \frac{\partial}{\partial u_i} Y_{l_1, \cdots, l_n} + \left(\frac{\mu_{u_i}}{\mu}\right) Y_{l_1, \cdots, l_n}, \end{aligned}$$

where all summations are extended over $p_1, p_2, \cdots, p_n$; $p_1 = 0, \cdots, l_1$; $\cdots$; $p_n = 0, \cdots, l_n$. Hence, by assumption (7),

$$\overline{Y_{l_1, \cdots, l_{i-1}, l_i+1, l_{i+1}, \cdots, l_n}} = \frac{\partial}{\partial u_i} \overline{Y_{l_1, \cdots, l_n}} + \overline{\left(\frac{\mu_{u_i}}{\mu}\right)} \overline{Y_{l_1, \cdots, l_n}}$$

$$= \frac{\partial}{\partial u_i}(\mu Y_{l_1, \cdots, l_n}) + \left[\left(\frac{\mu_{u_i}}{\mu}\right) - \frac{\mu_{u_i}}{\mu}\right]\mu Y_{l_1, \cdots, l_n}$$

$$= \mu \left\{ \frac{\partial}{\partial u_i} Y_{l_1, \cdots, l_n} + \left(\frac{\mu_{u_i}}{\mu}\right) Y_{l_1, \cdots, l_n} \right\}$$

$$= \mu Y_{l_1, \cdots, l_{i-1}, l_i+1, l_{i+1}, \cdots, l_n} \qquad (i = 1, \cdots, n),$$

which completes the induction proof.

The form of (5) shows that all the functions there listed would be independent except for the system (a). By means of (i) all y-derivatives are expressed linearly in terms of the primary derivatives. But (5) may be solved for y, y_{u_i}, etc., in terms of Y, Y_{u_i}, etc. Hence all semi-covariants (5) are functions of those which correspond to the primary derivatives, and of seminvariants.

A uniquely determined set of functions may also be obtained from the coefficients of $(\bar{a})$ by the substitutions

$$\mu_{u_i}/\mu = - y_{u_i}/y \qquad (i = 1, \cdots, n),$$

which are suggested by equations (2). This set of functions will be characterized by the relations

$$Y_{u_i} = 0 \qquad (i = 1, \cdots, n).$$

As a result of the uniqueness of this set, they are semi-covariants.

4. Application to algebraic semi-covariants

We may regard a pseudo-semi-canonical form of a completely integrable system of linear homogeneous partial differential equations as a formal analogue of the reduced binary form, and of the semi-canonical form of the linear homogeneous ordinary differential equation. Indeed, the suggestion contained in the last paragraph of section 3 may be carried out in the theory of binary forms.

When the binary p-ic*

$$f = a_0 x^p + pa_1 x^{p-1} y + \cdots + a_p y^p$$

is subjected to the transformation

$$x = \xi + n\eta, \qquad y = \eta,$$

the variables, $x^p, x^{p-1} y, \cdots, y^p$, are changed in accordance with the equations

$$(8) \qquad x^p = (\xi + n\eta)^p, \qquad x^{p-1} y = (\xi + n\eta)^{p-1}\eta, \qquad \cdots, \qquad y^p = \eta^p,$$

while the new coefficients are expressed in terms of the old by means of the equations

$$(9) \qquad a_0' = a_0, \qquad a_1' = a_1 + na_0, \qquad a_2' = a_2 + 2na_1 + n^2 a_0, \qquad \text{etc.}$$

* Cf. Dickson, *Algebraic Invariants*, 1914, p. 47.

The substitution in (9) of $n = -\xi/\eta$, suggested by (8), will yield a set of independent semi-covariants.

5. Effect of the transformation of the independent variables upon the coefficients of (a)

Let the independent variables be transformed by means of the equations

$$\hat{u}_i = U_i(u_i) \qquad (i = 1, \cdots, n). \tag{10}$$

The effect upon the variables y, y_{u_i}, etc., is shown by the following equations:

$$\begin{aligned}
&y = \hat{y}, \qquad y_{u_i} = U'_i \hat{y}_{u_i}, \qquad y_{u_i u_i} = U'^2_i \hat{y}_{u_i u_i} + U''_i \hat{y}_{u_i}, \\
&y_{u_i u_j} = U'_i U'_j \hat{y}_{u_i u_j}, \qquad (i \neq j), \\
&y_{u_i u_i u_i} = U'^3_i \hat{y}_{u_i u_i u_i} + 3U'_i U''_i \hat{y}_{u_i u_i} + U'''_i \hat{y}_{u_i}, \\
&y_{u_i u_i u_j} = U'^2_i U'_j \hat{y}_{u_i u_i u_j} + U''_i U'_j \hat{y}_{u_i u_j}, \qquad (i \neq j), \\
&y_{u_i u_j u_k} = U'_i U'_j U'_k \hat{y}_{u_i u_j u_k}, \qquad (i \neq j \neq k), \text{ etc.,} \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&y_{l_1, \cdots, l_n} = \hat{y}_{l_1, \cdots, l_n} \cdot U'^{l_1}_1 \cdots U'^{l_n}_n \\
&+ \sum_{g=1}^{n} q_g \hat{y}_{l_1, \cdots, l_{g-1}, l_g - 1, l_{g+1}, \cdots, l_n} \cdot U'^{l_1}_1 \cdots U'^{l_{g-1}}_{g-1} U'^{l_g - 2}_g U'^{l_{g+1}}_{g+1} \\
&\qquad\qquad \cdots U'^{l_n}_n U''_g + \cdots,
\end{aligned} \tag{11}$$

where $\hat{y}_{u_i} = \partial \hat{y}/\partial \hat{u}_i$, etc., and the q_g are integers (including zero). An easy induction suffices to show the correctness of the last expression of (11).

It is obvious from (11) that when the expressions there given for y, y_{u_i}, etc., are substituted in (a), and the resulting equations collected in a system $(\hat{a})$ of the same type, a given y-derivative may give increments only to those terms of its equation whose y-derivatives yield the given y-derivative by differentiation. For example, the new coefficient of $\hat{y}_{u_i u_j}$ in any equation of $(\hat{a})$ will be due to the old $y_{u_i u_j}, y_{u_i u_i u_j}$, etc., provided these derivatives are present in this equation. In this respect, the effects of the transformations (1a) and (10) are similar. There is this difference to be noticed, however. The right member of any equation of (2) will contain *all* y-derivatives which yield by differentiation the derivative in the left member of this equation. Of equations (11), on the other hand, there is only one, namely the first, of which this is true. For example, in the expression for $y_{u_i u_j}$, $(i \neq j)$, there occurs none of the variables $\hat{y}, \hat{y}_{u_i}, \hat{y}_{u_j}$, all of which would appear in the corresponding equation of (2).

In order to show more explicitly the effect of the transformation (10) upon (a), let one of the equations of (a) be the following:*

$$
\begin{aligned}
y_{l_1+1,\, l_2,\, \cdots,\, l_n} &= \sum_{g=2}^{n} \alpha_g\, y_{l_1,\, \cdots,\, l_{g-1},\, l_g+1,\, l_{g+1},\, \cdots,\, l_n} + \beta y_{l_1,\, \cdots,\, l_n} + \cdots \\
&\quad + \sum_{g=1}^{n} \epsilon_g\, y_{m_1 \cdots m_{g-1},\, m_g+1,\, m_{g+1},\, \cdots,\, m_n} + \zeta y_{m_1,\, \cdots,\, m_n} + \cdots,
\end{aligned} \tag{12}
$$

where $\Sigma l_i > \Sigma m_i$; and where the β-term is typical of all first class terms, while the ζ-term typifies all terms of higher class whose coefficients will, under the transformation (10), receive additional terms only from the y-derivatives

$$
y_{m_1,\, \cdots,\, m_{g-1}\, m_g+1,\, m_{g+1},\, \cdots,\, m_n} \qquad (g = 1, \cdots, n),
$$

of order one higher than $y_{m_1,\, \cdots,\, m_n}$.

The last equation of (11) gives us, as the expression for the y-derivatives from the ϵ-terms:

$$
\begin{aligned}
& y_{m_1,\, \cdots,\, m_{g-1},\, m_g+1,\, m_{g+1},\, \cdots,\, m_n} \\
&= \hat{y}_{m_1,\, \cdots,\, m_{g-1},\, m_g+1,\, m_{g+1},\, \cdots,\, m_n} \cdot U_1'^{m_1} \cdots U_{g-1}'^{m_{g-1}} U_g'^{m_g+1} U_{g+1}'^{m_{g+1}} \cdots U_n'^{m_n} \\
&\quad + q\hat{y}_{m_1,\, \cdots,\, m_n} \cdot U_1'^{m_1} \cdots U_{g-1}'^{m_{g-1}} U_g'^{m_g-1} U_{g+1}'^{m_{g+1}} \cdots U_n'^{m_n} U_g'' + \cdots,
\end{aligned} \tag{13}
$$

where we are assuming q, which cannot be negative, to be different from zero.

Suppose that one of the ϵ_g-terms receives an increment, by the transformation (10), from a y-derivative of higher order than that in the ϵ_g-term, say from $y_{l_1,\, \cdots,\, l_n}$. The expression for $y_{l_1,\, \cdots,\, l_n}$ will be found by successive differentiation of (13). But since the second term of the right member of (13) is a product, repeated differentiation will not make the $y_{m_1,\, \cdots,\, m_n}$ term disappear unless at least one of the U_i' and its derivatives are missing from the factor

$$
U_1'^{m_1} \cdots U_{g-1}'^{m_{g-1}} U_g'^{m_g-1} U_{g+1}'^{m_{g+1}} \cdots U_n'^{m_n} U_g''.
$$

Equations (11) show that this would mean that U_i' would also be absent from the coefficient of every y-derivative in the complete expression (13). Therefore, in forming out of (13) the expression for $y_{l_1,\, \cdots,\, l_n}$, not only $y_{m_1,\, \cdots,\, m_n}$, but $y_{m_1,\, \cdots,\, m_{g-1},\, m_g+1,\, m_{g+1},\, \cdots,\, m_n}$ as well, disappears. Hence, if an ϵ_g-term receives an increment from a higher order derivative in its equation, the ζ-term will likewise receive an increment from the same higher order derivative. But this is contrary to the assumption that the ζ-term receives increments only from the derivatives in the ϵ_g-terms. Therefore the ϵ_g are unaltered, except for a factor, by the transformation (10).

When the equations (11) are substituted in (12), we obtain the equation

$$
\hat{y}_{l_1+1,\, l_2,\, \cdots,\, l_n} \cdot U_1'^{l_1+1} U_2'^{l_2} \cdots U_n'^{l_n} + q_1 \hat{y}_{l_1,\, \cdots,\, l_n} \cdot U_1'^{l_1-1} U_2'^{l_2} \cdots U_n'^{l_n} U_1'' + \cdots
$$

* It is not material to the argument which one of the derivatives of order $1 + \Sigma l_i$ is the left member.

$$= \sum_{g=2}^{n} \alpha_g (\hat{\vartheta}_{l_1, \cdots, l_{g-1}, l_g+1, l_{g+1}, \cdots, l_n} \cdot U_1'^{l_1} \cdots U_{g-1}'^{l_{g-1}} U_g'^{l_g+1} U_{g+1}'^{l_{g+1}} \cdots U_n'^{l_n}$$

$$+ q_g \hat{\vartheta}_{l_1, \cdots, l_n} \cdot U_1'^{l_1} \cdots U_{g-1}'^{l_{g-1}} U_g'^{l_g-1} U_{g+1}'^{l_{g+1}} \cdots U_n'^{l_n} U_g'') + \cdots$$

$$+ \beta \hat{\vartheta}_{l_1, \cdots, l_n} \cdot U_1'^{l_1} \cdots U_n'^{l_n} + \cdots$$

$$+ \sum_{g=1}^{n} \epsilon_g (\hat{\vartheta}_{m_1, \cdots, m_{g-1}, m_g+1, m_{g+1}, \cdots, m_n} \cdot U_1'^{m_1} \cdot \quad U_{g-1}'^{m_{g-1}} U_g'^{m_g+1} U_{g+1}'^{m_{g+1}} \cdots U_n'^{m_n}$$

$$+ q_g' \hat{\vartheta}_{m_1, \cdots, m_n} \cdot U_1'^{m_1} \cdots U_{g-1}'^{m_{g-1}} U_g'^{m_g-1} U_{g+1}'^{m_{g+1}} \cdot U_n'^{m_n} U_g'') + \cdots$$

$$+ \zeta \hat{\vartheta}_{m_1, \cdots, m_n} \cdot U_1'^{m_1} \cdots U_n'^{m_n} + \cdots .$$

Upon dividing by $U_1'^{l_1+1} U_2'^{l_2} \cdots U_n'^{l_n}$, and transposing, we see that

$$\hat{\alpha}_g = \frac{U_g'}{U_1'} \alpha_g, \qquad \hat{\beta} = \frac{1}{U_1'} \left(\beta - q_1 \eta_1 + \sum_{g=2}^{n} q_g \alpha_g \eta_g \right),$$

(14)
$$\hat{\epsilon}_g = \frac{U_1'^{m_1} \cdots U_{g-1}'^{m_{g-1}} U_g'^{m_g+1} U_{g+1}'^{m_{g+1}} \cdots U_n'^{m_n}}{U_1'^{l_1+1} U_2'^{l_2} \cdots U_n'^{l_n}} \cdot \epsilon_g,$$

$$\hat{\zeta} = \frac{U_1'^{m_1} \cdots U_n'^{m_n}}{U_1'^{l_1+1} U_2'^{l_2} \cdots U_n'^{l_n}} \left(\zeta + \sum_{g=1}^{n} q_g' \epsilon_g \eta_g \right),$$

where

$$\eta_g \equiv U_g'' / U_g'.$$

All zeroth class coefficients, therefore, are unaltered, except for a factor, by the transformation (10). Since they are known to be unchanged by the transformation (1a), they are relative invariants.

If we write the $\hat{\beta}$ expression as

(15)
$$\hat{\beta} = \frac{1}{U_1'} \left(\beta + \sum_{g=1}^{n} q_g \alpha_g \eta_g \right) \qquad (\alpha_1 \equiv -1),$$

we may take this as typical (except for the factor), not only of all first class coefficients of $(\hat{a})$, but also of all coefficients, of whatever class, which are altered (factors disregarded) only by linear combinations of η_i $(i = 1, \cdots, n)$. In all such cases, the coefficients in these linear combinations are unchanged, except for a factor, by the transformation (10).

Consider any pseudo-semi-canonical form (A) of (a), characterized by the relations $B_g = 0$ $(g = 1, \cdots, n)$. We wish to observe the effect of (10) upon its coefficients. This may be done by direct substitution, but it is desirable for our purpose to indicate a different method.

The coefficients, $\bar{\beta}_g$, of $(\bar{a})$, whose vanishing characterizes the pseudo-semi-canonical form, may or may not be unaltered by the transformation (10). In the former case, the pseudo-semi-canonical form will be preserved, and equations (14), which describe the effect of (10) upon the original coefficients of (a), will serve also to indicate its effect upon the coefficients of the pseudo-semi-canonical form (A). We need only read A for α, B for β, etc.

In the latter case, the pseudo-semi-canonical form is violated, and must be restored by a second application of the transformation ($1a$). Equation (15) shows that the undesired increments, which must be removed from the $\hat{B}_g$, are at worst linear combinations of $\eta_1, \cdots, \eta_n$ with relative invariant multipliers. In this second application of ($1a$) to the form $(\hat{A})$, $\mu_{u_1}/\mu, \cdots, \mu_{u_n}/\mu$, must be replaced by linear combinations of the kind just described.

We recall that the coefficients of $(\bar{a})$ are equal to the corresponding coefficients of (a) plus linear combinations of μ-fractions with multipliers taken from among the original coefficients of (a). Hence, since we are applying ($1a$) to a form of (a) whose coefficients are seminvariants, we see that the coefficients of $(\hat{A})$ must, certain factors neglected, equal the corresponding coefficients of (A) plus functions of seminvariants and of the $\eta_1, \cdots, \eta_n$ and their derivatives.

The foregoing paragraphs throw some light on a question which arose in the author's previous paper.* Certain pseudo-semi-canonical forms in the special cases there discussed, not only yielded simpler seminvariants than did the classical process, but also produced a relatively large number of seminvariants which were relative invariants as well.

Equations (11) show that the coefficients of the primary derivative y will be unaltered by the transformation (10), except for a factor. For this reason, the system $(\hat{a})$ is usually much simpler than $(\hat{A})$. However, when a pseudo-semi-canonical form (A) has been used which is not violated by the transformation (10), the systems $(\hat{A})$ and $(\hat{a})$ are identical, and we may be sure that at least the coefficients of y in (A) will be relative invariants.

6. Pseudo-canonical forms and invariants

Equation (15), factors neglected, is typical of all coefficients of $(\hat{A})$ which have as increments only linear combinations of $\eta_1, \cdots, \eta_n$. Let us choose† n such coefficients and form the equations $\hat{B}_g = 0, (g = 1, \cdots, n)$. Assuming that the B_g have been so chosen as to permit, we solve the equations for $\eta_1, \cdots, \eta_n$, regarding these as independent unknowns. When these solutions, denoted by $(\eta_1), \cdots, (\eta_n)$, are substituted for the η's throughout $(\hat{A})$, we obtain (formally) a form $(\mathfrak{A})$ of (A), whose coefficients are unique except for certain factors $U_1'^{m_1} \cdots U_n'^{m_n}$. We shall call this form, which is characterized by the relations $\mathfrak{B}_g = 0$ $(g = 1, \cdots, n)$, a *pseudo-canonical form* of (a), and denote its coefficients by German capitals. By an argument

* A. L. Nelson, l. c. Cf. section 6.

† Cf. the discussion in section 8.

analogous to that used in section 2, we are able to prove that these coefficients are unchanged, factors neglected, by the transformation (10). Since, in addition, the coefficients of $(\hat{A})$ are functions of seminvariants and of $\eta_1, \cdots, \eta_n$, and the $(\eta_1), \cdots, (\eta_n)$ are seminvariants, it follows that *the coefficients of* $(\mathfrak{A})$ *are relative invariants.*

The invariants thus formed are in number $2n$ less than the total number of original coefficients of (a). That is to say, our set is n short of the number required for a fundamental set. Those we have are evidently independent, and it remains to indicate how n more may be formed, independent of each other and of those already obtained. These supplementary invariants may be found by a device exactly analogous to that employed by Wilczynski.*

In section 2 it was proved that

$$\left(\overline{\frac{\mu_{u_i}}{\mu}}\right) = \left(\frac{\mu_{u_i}}{\mu}\right) - \frac{\mu_{u_i}}{\mu}.$$

In an exactly similar manner, it may be shown that, except for a factor,

$$(\hat{\eta}_i) = (\eta_i) - \eta_i \qquad (i = 1, \cdots, n). \tag{16}$$

The second equation of (14) shows that $\alpha_g(\eta_g)$ must be transformed by (10) in accordance with the equations

$$\hat{\alpha}_g(\hat{\eta}_g) = \frac{1}{U_1'}[\alpha_g(\eta_g) + \cdots] \qquad (g = 1, \cdots, n).$$

Substitution of the first equation of (14) gives

$$(\hat{\eta}_g) = \frac{1}{U_g'}[(\eta_g) + \cdots], \tag{17}$$

for all (η_g) which arise from first class coefficients. Similarly, the third and fourth equations of (14) prove that for all other (η_g), the same factor, $1/U_g'$, occurs. Hence, combining (16) and (17), we see that the complete expression for $(\hat{\eta}_i)$ is

$$(\hat{\eta}_i) = \frac{1}{U_1'}[(\eta_i) - \eta_i] \qquad (i = 1, \cdots, n).$$

Choose any other relative invariant, θ, where

$$\hat{\theta} = U_1'^{l_1} \cdots U_n'^{l_n}\theta.$$

By differentiation, we obtain

$$\hat{\theta}_{u_i} = \frac{1}{U_1'}\frac{\partial}{\partial u_i}[U_1'^{l_1} \cdots U_n'^{l_n}\theta]$$

* *One-parameter Families*, etc., l. c. Cf. equations (25).

$$= \frac{1}{U_i'}[U_1'^{l_1} \cdots U_n'^{l_n} \theta_{u_i} + l_i U_1'^{l_1} \cdots U_{i-1}'^{l_{i-1}} U_i'^{l_i - 1} U_{i+1}'^{l_{i+1}} \cdots U_n'^{l_n} U_i'' \theta] \qquad (i = 1, \cdots, n),$$

so that

$$\frac{\hat{\theta}_{u_i}}{\hat{\theta}} = \frac{1}{U_1'}\left[\frac{\theta_{u_i}}{\theta} + l_i \eta_i\right] \qquad (i = 1, \cdots, n).$$

The functions

$$l_i(\eta_i) + \theta_{u_i}/\theta \qquad (i = 1, \cdots, n),$$

will be n new relative invariants, evidently independent of each other. They are also independent of the invariants which are coefficients of the pseudo-canonical form, because none of these coefficients possessed any of the (η_i) as leaders.

We recall here general principles underlying the formation of the coefficients of the pseudo-canonical form $(\mathfrak{A})$. Neglecting factors, the zeroth class coefficients of $(\mathfrak{A})$ are equal to the corresponding coefficients of the pseudo-semi-canonical form (A) from which $(\mathfrak{A})$ was formed. The non-vanishing coefficients of the first class are linear combinations of first class seminvariant coefficients of (A), the coefficients in these linear combinations being numerical multiples of zeroth class invariants. Each of the invariants of any particular class higher than the first, is equal to the corresponding seminvariant coefficient of (A) plus combinations of lower class seminvariant coefficients.

Hence, in the set of invariants, including the n invariants

$$l_i(\eta_i) + \theta_{u_i}/\theta \qquad (i = 1, \cdots, n), \tag{18}$$

we may solve for the corresponding seminvariant leaders. As a result, any invariant whatever of the original completely integrable system (a), being known to be a function of the fundamental seminvariants and their derivatives, is seen to be a function of the invariant coefficients of $(\mathfrak{A})$ and of the invariants (18). Therefore *these invariants are a fundamental set.*

7. Covariants

The semi-covariants (5) may be regarded as the variables of the pseudo-semi-canonical form (A). Under the transformation (10), they will be altered in accordance with equations (11), reading Y, Y_{u_i}, etc., for y, y_{u_i}, etc. We may solve these equations, so as to express $\hat{Y}$, $\hat{Y}_{u_i}$, etc., in terms of Y, Y_{u_i}, etc.

However, if the pseudo-semi-canonical form is violated by the transformation (10), it must be restored by a transformation of the type (1a). In other words, equations (2), with $\mu_{u_1}/\mu, \cdots, \mu_{u_n}/\mu$ so chosen as to put the system $(\hat{A})$ in the pseudo-semi-canonical form again, must in such cases follow the transformation (10), in order to give in final form the effect of (10) upon the semi-covariants (5). This will give us expressions for $\hat{Y}$, $\hat{Y}_{u_i}$, etc., as functions of the semi-covariants (5), the seminvariant coefficients of (A), and of $U_1', \cdots, U_n', \eta_1, \cdots, \eta_n$.

The substitutions $\eta_i = (\eta_i)$ $(i = 1, \cdots, n)$, which produce the pseudo-canonical form $(\mathfrak{A})$ of (A), will also give a unique set of equations which express $\hat{Y}$, $\hat{Y}_{u_i}$, etc., in terms of Y, Y_{u_i}, etc. Neglecting the factors $U_1'^{l_1}, \cdots, U_n'^{l_n}$, let us call these expressions $\mathfrak{Y}$, $\mathfrak{Y}_{u_i}$, etc. *They are relative covariants*, and are characterized as a set by the relations $\mathfrak{B}_g = 0$ $(g = 1, \cdots, n)$, which are characteristic of the pseudo-canonical form $(\mathfrak{A})$.

Those of the $\mathfrak{Y}$, $\mathfrak{Y}_{u_i}$, etc., up to a certain order, which is determined by the left members of the completely integrable system (a), are evidently independent. *All covariants are functions of these independent ones and of invariants.*

It is sufficient merely to remark that fundamental sets of simultaneous invariants and seminvariants of sets of completely integrable systems of partial differential equations may be constructed by the method outlined in the preceding sections. Examples of such sets, of which the simultaneous invariants might be of great interest, are suggested by two theorems of Koenigs* concerning the perspective plane nets of the asymptotic lines and of conjugate systems of curves on curved surfaces. Further examples are furnished by the Laplace suite, as applied to plane nets,† to conjugate nets of curves on surfaces,‡ and to congruences of straight lines.§

8. Concerning the possibility of securing a pseudo-canonical form

It must be emphasized that the "reduction" of a completely integrable system (a) to a pseudo-semi-canonical form is possible only when there are present n first class coefficients of $(\bar{a})$ of the type (3). Moreover, even after such a pseudo-semi-canonical form (A) has been obtained, the further "reduction" of (A) to a pseudo-canonical form can be accomplished only when we have n additional coefficients of $(\hat{A})$, of the first and higher classes, of the type (15), not corresponding to the n coefficients of $(\bar{a})$ previously used.

It would be very desirable, therefore, to show that the situation just described always arises. Unfortunately, this very general conclusion cannot be established. On the contrary, an important example is at hand to prove the impossibility of such a conclusion. We are, however, able to show that for a large class of completely integrable systems, the method with which this paper has concerned itself, may be used.

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* Paris Comptes Rendus, vol. 114 (1892), pp. 55–57; p. 728.

† E. J. Wilczynski, *One-parameter Families*, etc., l. c., section 4. See also A. L. Nelson, *Quasi-periodicity of Asymptotic Plane Nets*, Bulletin of the American Mathematical Society, vol. 22 (1916), pp. 445–455.

‡ G. M. Green, *Projective Differential Geometry of One-parameter Families of Space Curves*, etc., l. c., section 6.

§ E. J. Wilczynski, *Sur la Théorie Générale des Congruences*, l. c.

Suppose that the highest order derivative occurring in the completely integrable system is of order k, and that all kth order derivatives of y are present. For the purpose of our discussion, it is immaterial whether all the kth order derivatives are the left members of the equations of the system, or some are primary derivatives. There are in all*

$$H_k^n = C_k^{n+k-1} = \frac{n(n+1)\cdots(n+k-1)}{k!}$$

of these partial derivatives of order k. Among the cross-derivatives of this order will be a certain number of the type

(19) $$y_{l_1, \cdots, l_{r-1}, 1, l_{r+1}, \cdots, l_{s-1}, 1, l_{s+1}, \cdots, l_{t-1}, 1, l_{t+1}, \cdots, l_n},$$

which have been obtained by one and but one differentiation with respect to certain of the independent variables, u_r, u_s, u_t, etc. Each such derivative must yield an increment to the coefficient of every derivative of lower order out of which (19) comes by differentiation. In particular, the coefficients of the first class derivatives

(20) $$y_{l_1, \cdots, l_{r-1}, 0, l_{r+1}, \cdots, l_n}, \; y_{l_1, \cdots, l_{s-1}, 0, l_{s+1}, \cdots, l_n}, \; y_{l_1, \cdots, l_{t-1}, 0, l_{t+1}, \cdots, l_n}, \text{ etc.},$$

in the system $(\bar{a})$, will receive, respectively, the increments

(21) $$\mu_{u_r}/\mu, \; \mu_{u_s}/\mu, \; \mu_{u_t}/\mu, \text{ etc.}$$

Let us try to count the derivatives of the type (19), listing them under n heads, according as u_1, u_2, $\cdots$, or u_n, is the independent variable which is represented just once in (19). Except for duplications there would be $n \cdot H_{k-1}^{n-1}$ of these derivatives. However, when a certain derivative would be listed under a number of these heads, that single derivative will yield suitable increments to the same number of first class coefficients. Hence there are exactly $n \cdot H_{k-1}^{n-1}$ increments of type (21) to coefficients of first class derivatives (20) from k-th order derivatives (19).

For $n = 2$, this number is exactly 2, for all values of k. For $k = 2$, it is equal to $n(n-1)$. Moreover, H_{k-1}^{n-1} increases with either n or k. Therefore, for $n > 1$, $k > 1$, there will be at least n coefficients of the first class of $(\bar{a})$, of the required type, which will serve to determine a pseudo-semi-canonical form. Equations (1) show that such a pseudo-semi-canonical form will be undisturbed by the transformation (10).

It must be remembered that, in general, the $n \cdot H_{k-1}^{k-1}$ coefficients we have considered are by no means the only possible ones for our purpose. Derivatives of type (19) may yield suitable increments to other coefficients than those

* Chrystal, *Algebra*, Part II, sec. 10, p. 10.

of type (20). Moreover, kth order derivatives of other types than (19) may give rise to proper increments.

Having obtained a pseudo-semi-canonical form (A), determined as outlined above, our next concern is with the system $(\hat{A})$. But, as has been remarked in section 5, it will suffice to discuss $(\hat{a})$. Consider the n "straight" derivatives

$$y_{u_j^k} \quad (j = 1, \cdots, n),$$

of order k, in the system (a). Each of these will give an increment, $-q_j \eta_j$, to the coefficient of $y_{u_j^{k-1}}$ in its equation. None of these straight derivatives have been used in the determination of the pseudo-semi-canonical form. Hence they may be made use of to secure a pseudo-canonical form.

In addition to the coefficients of $(\hat{A})$ suggested, we have available, in general, certain suitably incremented coefficients of $(\hat{A})$ which come from cross-derivatives. Indeed, the cross-derivatives (19) yield increments of the proper type to certain first and higher class coefficients of (A) which have not been used in the determination of the pseudo-semi-canonical form.

The above discussion shows that one or more pseudo-canonical forms may surely be obtained in completely integrable systems of the rather regular type assumed, *provided all the kth order derivatives are present.* Indeed, especially for larger values of n or k, we are assured a large freedom in the choice of the $2n$ coefficients whose vanishing characterizes a pseudo-canonical form. This indicates that pseudo-canonical forms may be obtained in a great number of less regular completely integrable systems.

In cases where some of the kth order derivatives are primary derivatives, however, one or more of them may be removed by a preliminary transformation of the type

$$\hat{u}_i = \phi_i(u_1, \cdots, u_n) \qquad (i = 1, \cdots, n),$$

which refers the configuration to a particular set of parametric curves. If a sufficient number of these primary derivatives are removed from all of the equations of the completely integrable system, the number of suitably incremented coefficients of $(\bar{a})$ or $(\hat{A})$ may become less than n, so that the pseudo-semi-canonical form, or the pseudo-canonical form, or both, cannot be obtained.

An example of such systems is the one used by Wilczynski* in the study of curved surfaces referred to their asymptotic lines. It is only possible, in this case, to obtain a pseudo-semi-canonical form, which coincides with his canonical form.

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* E. J. Wilczynski, *Projective Differential Geometry of Curved Surfaces*, l. c. Cf. equations (27), (37), (38) and (46).